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Taylor vortices between almost cylindrical boundaries

P.M. EAGLES and K. EAMES

Department of Mathematics, The City University, London EC1, England

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Summary

We consider a modified Taylor problem, with the fluid flowing between a rotating inner circular cylinder and an outer stationary surface whose radius is a constant plus a small and slowly varying function of the axial co-ordinate z. This variation is chosen in such a way that the flow is locally more unstable near $z = 0$ than near $z = \pm\infty$, so that Taylor vortices appear more readily near $z = 0$. The theory is developed to show how vortices of strength varying with z develop as the speed of rotation is increased through a critical value which is a perturbation of the classical value. Wave number changes in the axial direction are also calculated.

1. Introduction

There has been considerable interest recently in modifications of the classical Bénard and Taylor-vortex problems, with a view to possibly obtaining solutions to more physically realistic problems. In fact, there is now considerable literature and we shall not attempt a survey here but rather mention just those papers which are closely related to the problem tackled here.

Kelly and Pal [1] considered a small spacially periodic perturbation of the wall boundary conditions in the Bénard problem, obtaining a theory which smoothed out the abrupt bifurcation of classical theory. This work was followed by Eames [2] for the Taylor vortex problem, in which the outer boundary was perturbed in a way to be described in this present paper. Eagles [3] perturbed the lower wall in the Bénard problem, and Walton [4] made a different study by performing a perturbation of the temperature of the lower wall in the Bénard problem in a more general way. The present paper is based in part on Eames' [2] thesis.

The mathematical theory of the Bénard and Taylor problems is similar in many ways. Thus many of the results of Eagles [3] and Walton [4] have analogues in the Taylor problem. However, the results presented here are original and interesting in three respects. Firstly, the steady state slowly varying base flow for the Taylor problem is much more complicated. Secondly, the expansion for the disturbance is taken to a higher order, enabling us to find wave number changes inaccessible to Eagles [3] and Walton [4]. Thirdly, the detailed numerical calculations given in Sec. 7 provide data which is quite new and should be amenable to experimental verification.

Let r, θ, z be cylindrical polar co-ordinates. We consider the flow of viscous fluid of constant density ρ between an inner cylinder of radius $r = R_1$ rotating with angular velocity Ω_1 and an outer stationary boundary of radius $r = H(Z)$. Here ε is a small parameter and

$$Z = \varepsilon z \tag{1.1}$$

is the "slow" variable in the z-direction.

After trial and error a form for $H(Z)$ amenable to analysis was found to be

$$H(Z) = R_2 + \varepsilon^2 dF(Z)/2 \tag{1.2}$$

where

$$d = R_2 - R_1. \tag{1.3}$$

This is a particular case of a type of variation $1 + \mu F(\varepsilon z)$ considered by Walton [4]. This enabled us to approximate both the steady state flow and the Taylor-vortex-like flow over the whole range $-\infty < z < \infty$, without any of the singularities which occur for some other cases (e.g. for an outer radius $r = H(Z)$; see Walton [4]). The particular form of (1.2) was determined by the requirement that using the dimensionless variable x defined in (3.3) the outer radius was of the form $x = \frac{1}{2}(1 + \varepsilon^2 F(z))$ which was convenient for the analysis.

The parameters which specify the geometry and physics of the problem are the Taylor number

$$T = 2\Omega_1^2 R_1^2 d^3 / \left(\nu^2 \{R_1 + R_2\}\right) \tag{1.4}$$

where ν is the kinematic viscosity, and the radius ratio at $z = 0$

$$\eta = R_1/R_2. \tag{1.5}$$

It is also convenient to define a local Taylor number

$$T = 2\Omega_1^2 R_1^2 d_L^3 / \left(\nu^2 \{R_1 + R_{2L}\}\right) \tag{1.6}$$

where $d_L(Z)$ and $R_{2L}(Z)$ are the "local" gap width and outer radius respectively,

We shall here choose $F(z)$ in such a way that

(i) $F(Z)$ is even, (ii) $F'(\pm\infty) = 0$,
(iii) $F(Z) \leqslant 0$, (iv) $F(0) = 0$.

A typical example is $F(Z) = -\tanh^2(\omega Z)$ and a typical geometry is illustrated schematically in Fig. 1.

In such a case the local Taylor number is smaller near $z = \infty$ than at $z = 0$. If then we increase the Taylor number (1.4) very slowly we might expect instability of the laminar Couette-like flow (described in Sec. 2) to occur more readily at the centre (z near zero) than at the ends. This idea is of course oversimplified but it provides a guide. We must take account of the derivatives with respect to z, which will be small in our case, to obtain a satisfactory theory.

In the subsequent work we develop a consistent theory for the linear stability problem by considering a neutral disturbance based on the most unstable mode of conventional

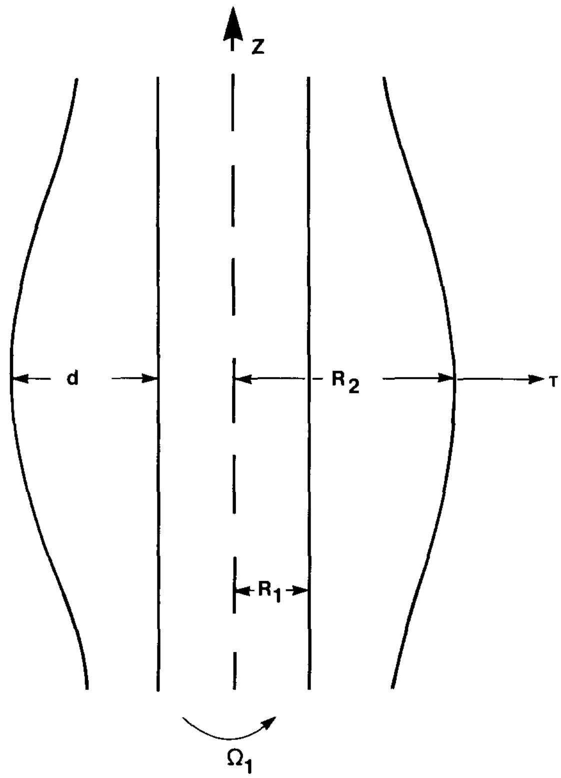

Figure 1. Schematic of the geometry.

(parallel walled) theory, with wavenumber λ_c, where the critical Taylor number for the parallel wall case is T_c.

The salient features of the results are:

(i) the increase in the critical Taylor number over the parallel walled case (see (7.1));
(ii) the rather rapid decay of the strength of the vortices as $|z|$ increases (see Figures 2 and 3) and
(iii) the change in wavenumber with z.

2. The slowly varying basic flow

The boundaries are $r = R_1$ and $r = R_2 + \frac{1}{2}\varepsilon^2(R_2 - R_1)F(Z)$ where $Z = \varepsilon z$. The inner cylinder is rotating with angular velocity Ω_1 while the outer boundary is at rest.

For $\varepsilon = 0$ we have the well-known Couette solution for the velocities in the r, θ and z directions respectively,

$$u = 0, \quad v = V_0(r) = Ar + B/r, \quad w = 0 \tag{2.1}$$

where $A = -\Omega_1 R_1^2/(R_2^2 - R_1^2)$ and $B = \Omega_1 R_1^2 R_2^2/(R_2^2 - R_1^2)$ and the pressure is given by

$$p = p_0(r) = \rho\left[A^2r^2/2 + 2AB\log r - \tfrac{1}{2}B^2/r^2 + C\right], \tag{2.2}$$

where C is a constant.

For the general case when $F(Z)$ is even and $F'(\pm\infty) = 0$ we denote the velocities and pressure by (u_s, v_s, w_s, p_s) and symmetry arguments suggest an expansion

$$u_s = \varepsilon^2 u_2(r, Z) + O(\varepsilon^4), \tag{2.3a}$$

$$v_s = V_0(r) + \varepsilon^2 v_2(r, Z) + O(\varepsilon^4), \tag{2.3b}$$

$$w_s = \varepsilon w_1(r, Z) + \varepsilon^3 W_3(r, Z) + O(\varepsilon^5), \tag{2.3c}$$

$$p_s = p_0(r) + \varepsilon^2 p_2(r, Z) + O(\varepsilon^4), \tag{2.3d}$$

where the functions $u_i(r, Z)$ and $v_i(r, Z)$ are even in Z and the functions $w_i(r, Z)$ are odd in Z. On substituting the above form of expansion into the Navier-Stokes and continuity equations we equate coefficients of powers of ε and solve the corresponding equations with the boundary conditions

$$u_s = v_s = w_s = 0 \quad \text{on} \quad r = R_2 + \varepsilon^2 dF(Z)/2,$$

$$u_s = w_s = 0 \quad \text{and} \quad v_s = \Omega_1 R_1 \quad \text{on} \quad r = R_1.$$

The boundary conditions on the outer wall for the u_i, v_i and w_i are dealt with by a Taylor expansion about $r = R_2$. An extra assumption is made that

$$u_s \to 0 \quad \text{and} \quad w_s \to 0 \quad \text{as} \quad Z \to \pm\infty.$$

This is consistent with $F'(Z) \to 0$ as $Z \to \pm\infty$ and a purely circumferential Couette flow at infinity.

Solving the resulting equations gives

$$w_1(r, Z) = 0, \quad u_2(r, Z) = 0, \tag{2.4}$$

$$v_2(r, Z) = \frac{\Omega_1 R_1^2 R_2 F(Z)}{(R_2^2 - R_1^2)(R_1 + R_2)}\left\{r - \frac{R_1^2}{r}\right\}, \tag{2.5}$$

$$w_3(r, Z) = \frac{\Omega_1^2 R_1^4 R_2 \mathrm{d}F/\mathrm{d}Z}{\nu(R_2^2 - R_1^2)^2(R_1 + R_2)} Q_3(r) \tag{2.6}$$

where

$$Q_3(r) = -\frac{1}{16}\left[r^4 + \frac{R_1^4\log(r/R_2) - R_2^4\log(r/R_1)}{\log(R_2/R_1)}\right] + \frac{R_1^2R_2^2}{2}\log\left(\frac{r}{R_1}\right)\log\left(\frac{r}{R_2}\right) + \frac{(R_1^2 + R_2^2)}{2}\left[(r^2 - R_1^2)\log(r/R_1)\right] + (b_0 - 1)\frac{(R_1^2 + R_2^2)}{2}\left[r^2 + \frac{R_1^2\log(r/R_2) - R_2^2\log(r/R_1)}{\log(R_2/R_1)}\right]. \tag{2.7}$$

The continuity equation must be used to find b_0 which along with the boundary conditions on w_3 gives the resulting equation

$$\int_{R_1}^{R_2} r w_3(r)\mathrm{d}r = 0. \tag{2.8}$$

It can be shown that b_0 depends only on $\eta = R_1/R_2$ and two cases were solved numerically with the results

$$b_0 = 0.3358 \quad \text{for} \quad \eta = 0.5, \tag{2.9}$$

$$b_0 = 0.0256 \quad \text{for} \quad \eta = 0.95. \tag{2.10}$$

We can then find

$$p_2(r, Z) = \frac{\rho\Omega_1^2 R_1^4 R_2 F(Z)}{\left(R_2^2 - R_1^2\right)^2 (R_1 + R_2)} P_2(r) \tag{2.11}$$

where

$$P_2(r) = -r^2 + 2\left(R_1^2 + R_2^2\right)(\log r/R_2) + \frac{R_1^2 R_2^2}{r^2} + 2b_0\left(R_1^2 + R_2^2\right). \tag{2.12}$$

This summary of the results is necessarily condensed. The work is not altogether trivial. Further details may be consulted in Eames [2].

3. The perturbation equations in matrix form

We follow Eagles [5] and write the disturbance equations in dimensionless matrix form as follows. We set $u = u'$, $v = v_s + v'$, $w = w_s + w'$, $p = p_s + p'$ and ignore terms of $O(\varepsilon^4)$. The primed variables are functions of r, z and t. The following constants are needed

$$d = R_2 - R_1, \quad R_0 = (R_1 + R_2)/2, \quad \delta = d/R_0, \quad \alpha = \delta R_1^2/(R_1 + R_2)^2. \tag{3.1}$$

The dimensionless disturbance velocities and pressure, defined as in Davey, DiPrima and Stuart [6] are u, v, w and p where

$$u' = -\nu u/\alpha d, \quad w' = -\nu w/\alpha d, \quad v' = \Omega_1 R_0 v/2, \quad p' = \left(-\nu^2\rho/\alpha d^2\right)p, \tag{3.2}$$

while the dimensionless co-ordinates x, ζ, τ and z^* are given by

$$r = R_0 + \mathrm{d}x, \quad z = \mathrm{d}\zeta, \quad t = d^2\tau/\nu, \quad Z = \mathrm{d}z^*. \tag{3.3}$$

We note especially that

$$z^* = \varepsilon\zeta \tag{3.4}$$

is the "slow" axial variable.

The Taylor number is here defined as

$$T = \Omega_1^2 R_1^2 d^3/\left(\nu^2 R_0\right) \tag{3.5}$$

and the radius ratio is

$$\eta = R_1/R_2. \tag{3.6}$$

These two parameters coupled with $f(z^*)$ specify the problem. The equations of motion also contain the dimensionless functions derived from the basic steady state flow,

$$G(x) = 1/(1+\delta x), \tag{3.7}$$

$$\sigma(x) = (1+\eta)/2 + (1-\eta)x, \tag{3.8}$$

$$\Omega_0(x) = -\frac{2\eta^2}{1-\eta^2} + \frac{8\eta^2}{(1+\eta)^2(1-\eta^2)}G^2(x), \tag{3.9}$$

$$\Omega_2(x) = \frac{2\eta^2}{(1-\eta^2)(1+\eta)} - \frac{8\eta^4}{(1+\eta)^3(1-\eta^2)}G^2(x), \tag{3.10}$$

$$V_2(x) = \frac{1}{4(1-\eta)G(x)} - \frac{\eta^2 G(x)}{(1-\eta^2)(1+\eta)}, \tag{3.11}$$

$$W_3(x) = \frac{\eta^2}{2(1-\eta)^3(1-\eta^2)^2}\left[\frac{1+\eta^2}{2\log\eta}\{(\sigma^2-\eta^2)\log\sigma\log\eta\right.$$
$$+(b_0-1)\left((\sigma^2-1)\log\eta+\left(1-\eta^2\right)\log\sigma\right)\}+\frac{\eta^2}{2}\log\sigma(\log\sigma/\eta)$$
$$\left.-\frac{1}{16\log\eta}\{(\sigma^4-1)\log\eta+\left(1-\eta^4\right)\log\sigma\}\right], \tag{3.12}$$

$$f(z^*) = F(Z). \tag{3.13}$$

Using the derivative with respect to x of the continuity equation to eliminate $\partial^2 u/\partial x^2$ from the first momentum equation we are able to write the disturbance equations in the following matrix form, where nonlinear terms have been neglected:

$$\frac{\partial \boldsymbol{U}}{\partial x} - \boldsymbol{A}\boldsymbol{U} - \boldsymbol{B}\frac{\partial \boldsymbol{U}}{\partial \tau} - \varepsilon^2 f(z^*)\boldsymbol{C}\boldsymbol{U} + \varepsilon^3\frac{\partial f}{\partial z^*}\boldsymbol{D}\boldsymbol{U} + \ldots = \boldsymbol{O}. \tag{3.14}$$

Here

$$\boldsymbol{U} = [p, v_0, w_0, u, v, w]^{\mathrm{tr}} \tag{3.14a}$$

and $v_0 = \partial v/\partial x$, $w_0 = \partial w/\partial x$. The matrices are given by

$$A = \begin{bmatrix} 0 & 0 & -\partial/\partial\zeta & \partial^2/\partial\zeta^2 & -T\Omega_0 & 0 \\ 0 & -\delta G & 0 & 0 & -\partial^2/\partial\zeta^2 + \delta^2 G^2 & 0 \\ \partial/\partial\zeta & 0 & -\delta G & 0 & 0 & -\partial^2/\partial\zeta^2 \\ 0 & 0 & 0 & -\delta G & 0 & -\partial/\partial\zeta \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{3.14b}$$

$$B = \begin{bmatrix} \boldsymbol{O} & \begin{matrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}, \tag{3.15}$$

$$C = \begin{bmatrix} \boldsymbol{O} & \begin{matrix} 0 & T\Omega_2 & 0 \\ \dfrac{1}{1+\eta} & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}, \tag{3.16}$$

$$D = \begin{bmatrix} \boldsymbol{O} & \begin{matrix} TW_3\dfrac{\partial}{\partial\zeta} & 0 & 0 \\ 0 & -TW_3\dfrac{\partial}{\partial\zeta} & V_2 \\ -T\dfrac{\mathrm{d}W_3}{\mathrm{d}x} & 0 & -TW_3\dfrac{\partial}{\partial\zeta} \end{matrix} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}. \tag{3.17}$$

The equation (3.14) has to be solved subject to the conditions that $\boldsymbol{U} = \boldsymbol{O}$ on the inner and outer boundaries. For the vector $\boldsymbol{U}$ of six components this is labelled

β_3: the last three components of $\boldsymbol{U}$ are zero at $x = 1/2$ and $x = (1/2)\{1 + \varepsilon^2 f(z^*)\}$. (3.18)

The additional boundary condition

$$\boldsymbol{U} \to \boldsymbol{O} \text{ as } z^* \to \pm\infty \tag{3.19}$$

is imposed. A less restrictive condition that $\boldsymbol{U}$ be bounded as $z^* \to \pm\infty$ is acceptable, but it has been found that the most unstable disturbance results from (3.17), see Eames [2], and Eagles [3]. We shall next search for the marginally stable linear solutions of (3.12) with $\partial \boldsymbol{U}/\partial\tau = \boldsymbol{O}$.

4. The expansion procedure for the linear problem

In the usual parallel wall theory there is a neutral curve in the (λ, T) plane on which the disturbance has zero growth rate, where the mode of disturbance has a factor $\sin \lambda\zeta$. The minimum point of this curve is denoted by (λ_c, T_c). In the present case we may expect a neighbouring solution and therefore we set

$$U = e^{i\lambda_c \zeta}\left[\boldsymbol{u}_1(x, z^*) + \varepsilon \boldsymbol{u}_2(x, z^*) + \varepsilon^2 \boldsymbol{u}_3(x, z^*) + \dots\right]$$
$$+ \text{complex conjugate.} \tag{4.1}$$

This is the appropriate expansion for the *linear theory*. We also set

$$T = T_c + \varepsilon T_1^* + \varepsilon^2 T_2^* + \dots \tag{4.2}$$

Substituting (4.1) and (4.2) into (3.14) we find

$$\mathfrak{L}^{(1)}(\boldsymbol{u}_i) = 0, \tag{4.3}$$

$$\mathfrak{L}^{(1)}(\boldsymbol{u}_2) = \boldsymbol{B}_{c1}^{(1)} \frac{\partial \boldsymbol{u}_1}{\partial z^*} + T_1^* \boldsymbol{A}_2 \boldsymbol{u}_1, \tag{4.4}$$

$$\mathfrak{L}^{(1)}(\boldsymbol{u}_3) = \boldsymbol{B}_{c1}^{(1)} \frac{\partial \boldsymbol{u}_2}{\partial z^*} + T_1^* \boldsymbol{A}_2 \boldsymbol{u}_2 + \boldsymbol{B}_{c2} \frac{\partial^2 \boldsymbol{u}_1}{\partial z^{*2}}$$
$$+ T_2^* \boldsymbol{A}_2 \boldsymbol{u}_i - f(z^*) \boldsymbol{C}_c \boldsymbol{u}_1, \tag{4.5}$$

$$\mathfrak{L}^{(1)}(\boldsymbol{u}_4) = \boldsymbol{B}_{c1}^{(1)} \frac{\partial \boldsymbol{u}_3}{\partial z^*} + T_1^* \boldsymbol{A}_2 \boldsymbol{u}_3 + \boldsymbol{B}_{c2} \frac{\partial^2 \boldsymbol{u}_2}{\partial z^{*2}}$$
$$+ T_2^* \boldsymbol{A}_2 \boldsymbol{u}_2 - f(z^*) \boldsymbol{C}_c \boldsymbol{u}_2 + T_3^* \boldsymbol{A}_2 \boldsymbol{u}_1$$
$$- f(z^*) T_1^* \boldsymbol{C}_{c1} \boldsymbol{u}_1 - \frac{\mathrm{d}f}{\mathrm{d}z^*} \boldsymbol{D}_c^{(1)} \boldsymbol{u}_1. \tag{4.6}$$

Here

$$\mathfrak{L}^{(p)} = \frac{\partial}{\partial x} - \boldsymbol{A}_c^{(p)}, \tag{4.7}$$

with $\boldsymbol{A}_c^{(p)}$ defined by

$$\{\boldsymbol{A}_c^{(p)} \text{ is obtained from } \boldsymbol{A} \text{ by replacing } T \text{ by } T_c \text{ and } \partial/\partial\zeta \text{ by } ip\lambda_c.\} \tag{4.8}$$

while the matrices are defined as follows:

$$B_{c1}^{(m)} = \begin{bmatrix} 0 & 0 & -1 & 2\,\mathrm{im}\,\lambda_c & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\,\mathrm{im}\,\lambda_c & 0 \\ 1 & 0 & 0 & 0 & 0 & -2\,\mathrm{im}\,\lambda_c \\ & & & 0 & 0 & -1 \\ & \boldsymbol{O} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{bmatrix}, \tag{4.9}$$

$$B_{c2} = -B,$$

$$C_{c1} = \begin{bmatrix} \boldsymbol{O} & 0 & \Omega_2 & 0 \\ & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \boldsymbol{O} & & \boldsymbol{O} & \end{bmatrix}, \tag{4.10}$$

$$D_c^{(m)} = \begin{bmatrix} \boldsymbol{O} & \mathrm{im}\,\lambda_c T_c W_3 & 0 & 0 \\ & 0 & -\mathrm{im}\,\lambda_c T_c W_3 & V_2 \\ & -T_c\,\mathrm{d}W_3/\mathrm{d}x & 0 & -\mathrm{im}\,\lambda_c T_c W_3 \\ \boldsymbol{O} & & \boldsymbol{O} & \end{bmatrix}, \tag{4.11}$$

$$A_2 = \begin{bmatrix} \boldsymbol{O} & 0 & -\Omega_0 & 0 \\ & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \boldsymbol{O} & & \boldsymbol{O} & \end{bmatrix} \tag{4.12}$$

and

$$C_c \text{ is } C \text{ with } T \text{ replaced by } T_c. \tag{4.13}$$

The boundary conditions on the outer wall $x = (1/2)\{1 + \varepsilon^2 f(z^*)\}$ are obtained by expansion about $x = 1/2$ and we obtain for $j = 4, 5, 6$

$$u_{ij}(\tfrac{1}{2}, z^*) = 0, \quad u_{3j}(\tfrac{1}{2}, z^*) + \tfrac{1}{2} f(z^*) u_{ijx}(\tfrac{1}{2}, z^*) = 0 \tag{4.14}$$

and

$$u_{2j}(\tfrac{1}{2}, z^*) = 0, \quad u_{4j}(\tfrac{1}{2}, z^*) + \tfrac{1}{2} f(z^*) u_{2jx}(\tfrac{1}{2}, z^*) = 0 \tag{4.15}$$

where

$$u_{kjx} = \frac{\partial u_{kj}}{\partial x}$$

and u_{kj} denotes the jth component of $\boldsymbol{u}_k$. The inner wall conditions give

$$u_{1j}(-\tfrac{1}{2}, z^*) = 0, \quad u_{2j}(-\tfrac{1}{2}, z^*) = 0, \quad u_{3j}(-\tfrac{1}{2}, z^*) = 0. \tag{4.16}$$

5. Solutions of the disturbance equations

We need to use the adjoint eigenfunction as defined in Eagles [5]. This is denoted by $\boldsymbol{f}_a$ where

$$\frac{\partial \boldsymbol{f}_a}{\partial x} + \{\boldsymbol{A}_c^{(1)}\}^{\text{tr}} \boldsymbol{f}_a = 0. \tag{5.1}$$

and where the first three components of $\boldsymbol{f}_a$ are zero at $x = \pm\frac{1}{2}$.

This is used to obtain conditions for ordinary differential equations of the form

$$\frac{\partial \boldsymbol{u}}{\partial x} - \boldsymbol{A}_c^{(1)} \boldsymbol{u} = \boldsymbol{R}(x) \tag{5.2}$$

to have solutions, by multiplying by $\boldsymbol{f}_a^{\text{tr}}$ and integrating from $-1/2$ to $+1/2$, using the boundary conditions.

The general solution of (4.3) together with its homogeneous boundary conditions, based on the most unstable eigenfunction $\boldsymbol{u}_{11}(\boldsymbol{x})$ is

$$\boldsymbol{u}_1 = \psi(z^*) \boldsymbol{u}_{11}(x). \tag{5.3}$$

Here $\boldsymbol{u}_{11}(x)$ is the critical eigenfunction of the standard straight walled problem with $\lambda = \lambda_c$ and $T = T_c$. The normalization is arbitrary, but here we took the second component of $\boldsymbol{u}_{11}(x)$ at $x = -1/2$ equal to 1.

A crucial result used in the following is that

$$\int_{-1/2}^{1/2} \boldsymbol{f}_a^{\text{tr}} \boldsymbol{B}_{c1}^{(1)} \boldsymbol{u}_{11} \mathrm{d}x = 0. \tag{5.4}$$

This can be deduced from the fact that (λ_c, T_c) is a minimum of the neutral curve for the parallel wall case, (see Eagles [2], Eames [3]).

Substituting (5.3) into (4.4), using the existence condition and (5.4) yields

$$T_1^* = 0.$$

We then see that

$$\boldsymbol{u}_2(x, z^*) = \frac{\mathrm{d}\psi}{\mathrm{d}z^*} \boldsymbol{g}_{21}(x) + S(z^*) \boldsymbol{u}_{11}(x)$$

where $\boldsymbol{g}_{21}$ satisfies

$$\mathcal{L}^{(1)}(\boldsymbol{g}_{21}) = \boldsymbol{B}_{c1}^{(1)} \boldsymbol{u}_{11};\ \beta_2 \tag{5.5}$$

with the boundary conditions being denoted by

$$\{\beta_2\text{: the last three components to be zero at } x = \pm\tfrac{1}{2}\} \tag{5.6}$$

and where $S(z^*)$ is at present unknown.

Proceeding in the same way we find that the condition for (4.5) to have a solution is

$$\frac{d^2\psi}{dz^{*2}} + \psi\left[\frac{T_2^*}{T_2} + af(z^*)\right] = 0 \tag{5.7}$$

where

$$a = \frac{\frac{1}{2}f_a^{\mathrm{tr}}(\frac{1}{2})u_{11x}(\frac{1}{2}) - \int_{-1/2}^{1/2} f_a^{\mathrm{tr}} C_c u_{11} \mathrm{d}x}{T_2\int_{-1/2}^{1/2} f_a^{\mathrm{tr}} A_2 u_{11} \mathrm{d}x} \tag{5.8}$$

and T_2 is the constant appearing in the parallel wall neutral curve expansion

$$T - T_c = T_2(\lambda - \lambda_c)^2 + \ldots \tag{5.9}$$

A general integral expression for T_2 is given in the Appendix. Numerical values of T_2 were obtained (see Eames [2]) as follows:

$$T_2 = 256.506 \quad \text{for} \quad \eta = 0.95, \tag{5.10}$$

$$T_2 = 440.282 \quad \text{for} \quad \eta = 0.5. \tag{5.11}$$

We note that a depends on η.

The amplitude equation (5.7) together with the boundary conditions

$$\psi \to 0 \quad \text{as} \quad z^* \to \pm\infty \tag{5.12}$$

will determine the possible values of T_2^*. This will be discussed later, but we note the only solutions for ψ are either odd or even when $f(z^*)$ is even.

The expansion may be continued with

$$u_3 = \psi f(z^*)h_{31}(x) + \psi g_{31}(x) + \frac{\mathrm{d}S}{\mathrm{d}z^*}g_{21}(x) + P(z^*)u_{11}(x) \tag{5.13}$$

where

$$\mathfrak{L}^{(1)}(h_{31}) = -a\left[\left(B_{c2}^{(1)}g_{21} + B_{c2}u_{11}\right) + C_c u_{11}\right] \tag{5.14}$$

and

$$\mathfrak{L}^{(1)}(g_{31}) = -\frac{T_2^*}{T_2}\left[B_{c1}^{(1)}g_{21} + B_{c2}u_{11} - T_2 A_2 u_{11}\right] \tag{5.15}$$

with boundary conditions

$$h_{31j}(-\tfrac{1}{2}) = g_{31j}(-\tfrac{1}{2}) = 0 \tag{5.16}$$

and

$$h_{31j}(\tfrac{1}{2}) = -\tfrac{1}{2}u_{11jx}(\tfrac{1}{2}), \; g_{31j}(\tfrac{1}{2}) = 0 \tag{5.17}$$

for $j = 4$, 5 and 6.

Using (5.7) and (5.17) we eventually find that the condition for (4.6) to have a solution is

$$\frac{d^2S}{dz^{*2}} + S(z^*)\left[\frac{T_2^*}{T_2} + af(z^*)\right] = r_1 f(z^*)\frac{d\psi}{dz^*} + r_2\psi\frac{df}{dz^*} + r_3\frac{d\psi}{dz^*} - \psi\frac{T_3^*}{T_2} \tag{5.18}$$

where r_1, r_2 and r_3 are constants. The boundary conditions are

$$S(z^*) \to 0 \quad \text{as} \quad z^* \to \pm\infty. \tag{5.19}$$

By multiplying (5.18) by $\psi(z^*)$ and integrating from $-\infty$ to ∞ and remembering that $\psi(z^*)$ is either odd or even we find $T_3^* = 0$.

The values of r_1, r_2 and r_3 are expressible in terms of integrals of known functions (see Appendix) and are purely imaginary.

It turns out that $S(z^*)$ is purely imaginary. It thus gives a *higher order correction to the wavenumber.*

In fact for a typical velocity component U_4 we can show that the wavenumber * is

$$N(U_4) = \lambda_c + \varepsilon^2\left[\frac{d}{dz^*}\left(\frac{S_i}{\psi}\right) + \frac{g_{214}^{(i)}}{u_{114}^{(r)}}\frac{d}{dz^*}\left(\frac{d\psi}{dz^*}/\psi\right)\right] + \dots \tag{5.20}$$

where $g_{214}^{(i)}$ is the imaginary part of g_{214} (its real part being zero), $u_{114}^{(r)}$ is the real part of u_{114} (its imaginary part being zero). Also $S = iS_i$ with S_i real and ψ is also real.

It should be noted that the wavenumber depends both on x and z^*. This will result physically in changes both in size and shape of the Taylor-vortex cells due to the varying radius of the outer boundary.

6. Solutions of the amplitude equations

We take as a specific example

$$f(z^*) = -\tanh^2\omega z^*. \tag{6.1}$$

We can find simple cases by choosing special values of the constant ω such that

$$\omega^2 = \frac{a}{j(j+1)} \tag{6.2}$$

for $j = 1, 2, 3\dots$, and a is the constant occurring in the amplitude equation (5.7).

The eigenvalues are then

$$T_2^* = aT_2\left[1 - \frac{(j-n)^2}{j(j+1)}\right] \tag{6.3}$$

* For an arbitrary complex function of Z the wavenumber may be defined as $\partial/\partial\zeta\{\arg f(Z)\}$.

where $n < j$, and the corresponding eigenfunctions are of the form

$$\psi_{nj} = A_{nj}\,\mathrm{sech}^{j-n}(\omega z^*)\,w_1(\tanh \omega z^*) \tag{6.4}$$

where $w_1(u)$ is a polynomial in u of degree n.

For example with $j = 1$ we have just one eigenvalue with $n = 0$. This is

$$T_2^* = \tfrac{1}{2} a T_2 \tag{6.5}$$

and the eigenfunction is

$$\psi_{01} = A\,\mathrm{sech}(\omega z^*). \tag{6.6}$$

With $j = 2$, there are two eigenvalues

$$T_2^* = \tfrac{1}{3} a T_2 \quad \text{and} \quad T_2^* = 5aT_2/6 \tag{6.7}$$

and corresponding eigenfunctions are $\psi_{02} = \mathrm{sech}^2(\omega z^*)$ and $\psi_{12} = \mathrm{sech}(\omega z^*)\tanh(\omega z^*)$. Some further details are given in Table 1.

We shall be concerned only with $n = 0$ because this gives the lowest value of T at which neutral stability occurs for a fixed ω (i.e. fixed j).

The solution for $S(z^*)$ is $S = iS_i$ where

$$S_i(z^*) = \tfrac{1}{2}\psi_{0j}\left[(r_{3i} - r_{1j})z^* + \frac{2r_{2i} + jr_{1i}}{(j+1)\omega}\tanh(\omega z^*)\right] \tag{6.8}$$

where r_{ki} is the imaginary part of r_k, the real part being zero in each case.

7. Numerical results and discussion for $\eta = 0.5$

The methods of computation are similar to those of Eagles [5] and need not be described again, except to say that numerous consistency checks were made by changing arbitrarily the scalings of the various functions involved and following through the results numerically and theoretically. Comparisons were made with earlier work wherever possible.

We concentrate here on presenting some detailed results for the case $\eta = 0.5, j = 1$ with

Table 1
Eigenvalues and eigensolutions for $\eta = 0.5$ and fixed j and n

j	n	ω	T_2^*	$\psi_{nj}(z^*)$
1	0	1.9354	1649.25	$A\,\mathrm{sech}\,z^*$
2	0	1.1174	1099.50	$A\,\mathrm{sech}^2\omega z^*$
	1		2748.76	$A\,\mathrm{sech}\,\omega z^*\,\tanh\,\omega z^*$
3	0	0.7901	824.63	$A\,\mathrm{sech}^3\omega z^*$
	1		2199.00	$A\,\mathrm{sech}^2\omega z^*\tanh\,\omega z^*$
	2		3023.63	$A\,\mathrm{sech}\,\omega z^*[\tanh^2\omega z^* - \frac{1}{5}]$

$f(z^*) = -\tanh^2 \omega z^*$ as described in Sec. 6. There is then only one isolated eigenvalue $T_2^* = \frac{1}{2} a T_2$ and the overall critical Taylor number is

$$T_{\text{crit}} = T_c + \varepsilon^2 T_2^* + \ldots$$

$$= 3099.8 + \varepsilon^2 1649.25 + \ldots, \qquad (7.1)$$

Thus the flow is more stable than the straight walled case with $\eta = 0.5$.

For $T > T_{\text{crit}}$ we expect Taylor-like vortices to appear with a positive growth rate. These will reach an equilibrium amplitude in practice due to the effects of the nonlinear terms. See Eagles [3], Eames [2] for details of this nonlinear theory.

But nevertheless for T just slightly greater than T_c the linear theory gives a good guide to the appearance of the vortices, and could be compared with experiment in a suitable apparatus.

The values of the constants needed were computed to be as follows

$$\lambda_c = 3.16242, \quad T_c = 3099.78, \quad T_2 = 440.2819,$$

$$a = 7.4918, \quad r_{1i} = -1.4179, \quad r_{2i} = 1.3302,$$

$$r_{3i} = 0.8503, \quad \omega = 1.9354.$$

Using the theory of Sections 5 and 6 we computed the solution to order ε for both $\varepsilon = 0.1$ (Fig. 2) and $\varepsilon = 0.5$ (Fig. 3), where the fourth component of U is plotted against z^*, this being the x-component of velocity. Results for other components are similar. Table 2 will enable the reader to compute solutions for other values of ε.

The main feature is the comparatively rapid decay of the strength of the vortices as $|z^*|$

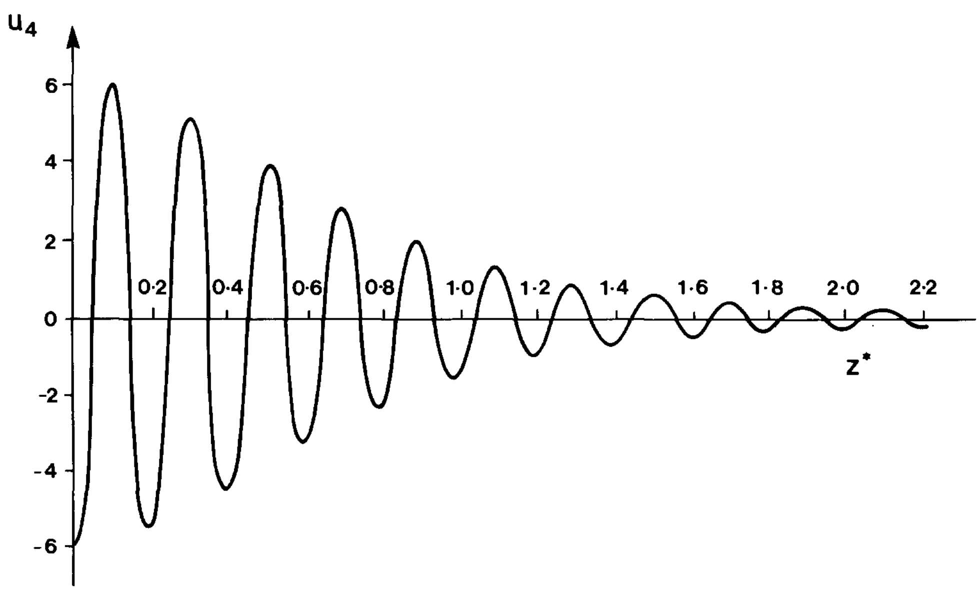

Figure 2. Velocity U_4 for $\eta = 0.5$, $x = 0, j = 1$, $\varepsilon = 0.1$.

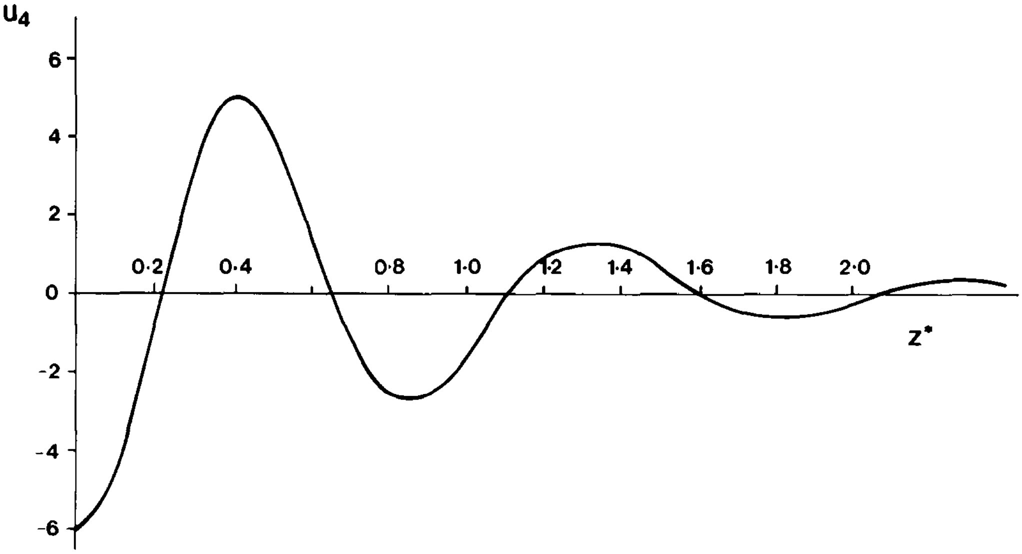

Figure 3. Velocity U_4 for $\eta = 0.5$, $x = 0$, $j = 1$, $\varepsilon = 0.5$.

increases, due primarily to the multiplying factor of $\psi(z^*) = \operatorname{sech} \omega z^*$ in the first term of the expansion. A less obvious feature is the change in axial wavenumber both with x and z^*. Compared with the parallel wall solution with $\eta = 0.5$ the size of the vortices near the centre ($z^* = 0$) are smaller and they also take a slightly different shape. For the case $j = 1$, $\varepsilon = 0.5$ the wavelength is about 10% less near $z^* = 0$ than the parallel walled wavelength. This effect should be distinguishable experimentally.

In order to emphasise this effect we show in Fig. 4 a plot of the wavenumber for U_4 as defined in (5.20) plotted against η_L, where η_L is the "local" radius ratio, that is

$$\eta_L = \frac{R_1}{R_2 + \varepsilon^2 d f(z^*)/2} \tag{7.2}$$

which in this case increases from 0.5 at $z^* = 0$ to 0.532 at $z^* = \infty$.

Table 2
Velocity component functions for $\eta = 0.5$. Here i denotes $\sqrt{-1}$, $u_{11,4}$ is the fourth component of $\boldsymbol{u}_{11}$ and $g_{21,4}$ is the fourth component of g_{21}

x	$u_{11,4}$	$i\,g_{21,4}$
−0.5	0.0	0.0
−0.4	−1.196	−0.463
−0.3	−3.410	1.226
−0.2	−5.295	1.779
−0.1	−6.238	1.971
0.0	−6.121	1.839
0.1	−5.128	1.487
0.2	−3.598	1.029
0.3	−1.935	0.561
0.4	−0.576	0.174
0.5	0.0	0.0

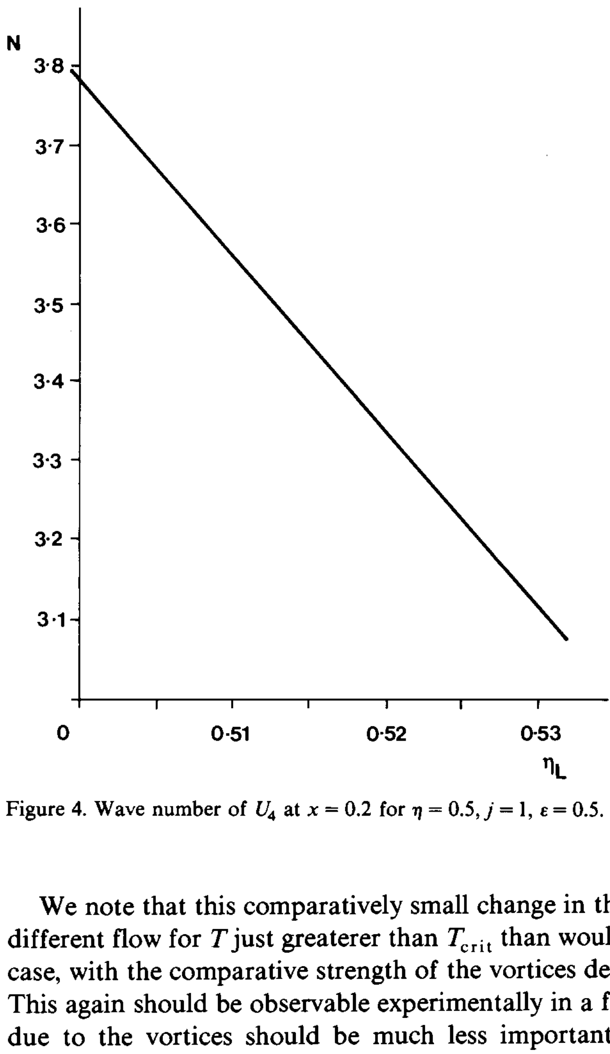

Figure 4. Wave number of U_4 at $x = 0.2$ for $\eta = 0.5, j = 1, \varepsilon = 0.5$.

We note that this comparatively small change in the radius ratio produces a drastically different flow for T just greaterer than T_{crit} than would be obtained for the straight-walled case, with the comparative strength of the vortices decaying rapidly with z^* (as in Fig. 3). This again should be observable experimentally in a finite length apparatus, as end effects due to the vortices should be much less important. If for $T < T_{\text{crit}}$ one could obtain experimentally Couette circumferential flow over a range of z^* of say -10 to 10 then the effect of increasing T should be to the type of flow shown in Fig. 3. The nonlinear theory of Eames [2] shows that the pattern of Fig. 4 is retained essentially as T increases slightly above T_{crit}, with the amplitude increasing proportionally to $\sqrt{T - T_{\text{crit}}}$.

In Figs. 5 and 6 we plot the stream lines for the vortices of Fig. 3. These were computed by using the Stokes stream function Φ, obtained from the velocity expansion, and the plots are of curves $\Phi = c$ for various values of c, but it should be remembered that the normalization adopted here for our linear theory is arbitrary.

There is one other point to be made. Both Eames [2], and Eagles [3] show that the choice of the type of expansion made here, starting with $e^{i\lambda_c \zeta} U_1(x, z^*)$, is important. One cannot obtain a neutral solution with a different basic wavelength, for the expansion becomes inconsistent if not based on λ_c, T_c. Thus the linear neutral solution, and the Taylor-vortex-like flow based on it, appears to be unique in this sense, unlike the results of

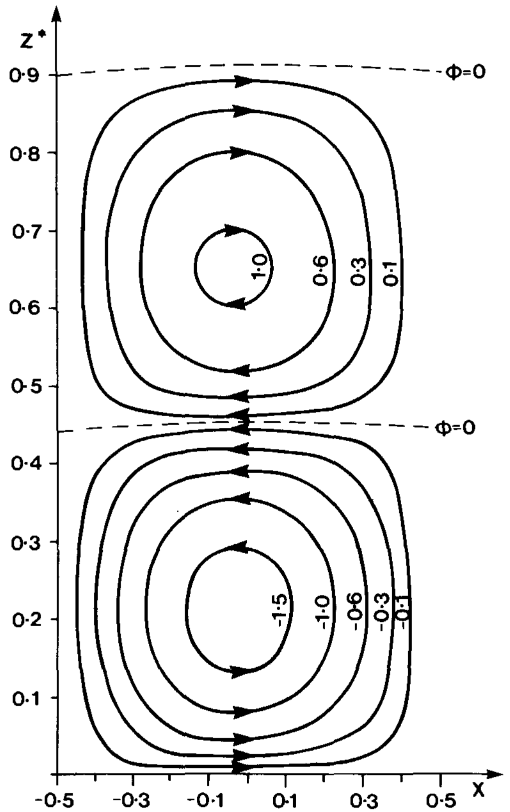

Figure 5. First and second vortex streamlines for $\eta = 0.5, j = 1$ and $\varepsilon = 0.5$.

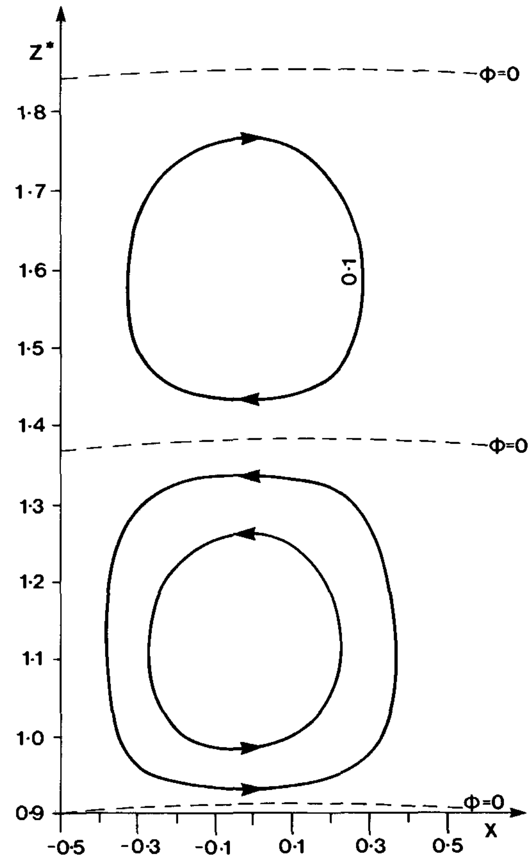

Figure 6. Third and fourth vortex streamlines for $\eta = 0.5, j = 1$ and $\varepsilon = 0.5$.

parallel wall theory. As T increases through T_{crit} only one unique Taylor vortex type flow would be possible according to the present theory.

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Appendix

Formulae for T_2, r_1, r_2 and r_3

The required formulae are

$$T_2 = \frac{\int_{-1/2}^{1/2} \boldsymbol{f}_a^{\text{tr}}\left(\boldsymbol{B}_{c1}^{(1)}\boldsymbol{g}_{21} - \boldsymbol{B}_{c2}\boldsymbol{u}_{11}\right)\mathrm{d}x}{\int_{-1/2}^{1/2} \boldsymbol{f}_a^{\text{tr}}\boldsymbol{A}_2\boldsymbol{u}_{11}\mathrm{d}x},$$

$$-r_1 = \frac{\frac{1}{2}\left[\boldsymbol{f}_a^{\text{tr}}\boldsymbol{g}_{21}\right]_{x=1/2} + \int_{-1/2}^{1/2} \boldsymbol{f}_a^{\text{tr}}\left[\boldsymbol{B}_{c1}\boldsymbol{h}_{31} - \boldsymbol{C}_c\boldsymbol{g}_{21} - a\boldsymbol{B}_{c2}\boldsymbol{g}_{21}\right]\mathrm{d}x}{T_2\int_{-1/2}^{1/2} \boldsymbol{f}_a^{\text{tr}}\boldsymbol{A}_2\boldsymbol{u}_{11}\mathrm{d}x},$$

$$-r_2 = \frac{\int_{-1/2}^{1/2} \boldsymbol{f}_a^{\text{tr}}\left[\boldsymbol{B}_{c1}^{(1)}\boldsymbol{g}_{31} + T_2^*\boldsymbol{A}_2\boldsymbol{g}_{21} - (T_2^*/T_2)\boldsymbol{B}_{c2}\boldsymbol{g}_{21}\right]\mathrm{d}x}{T_2\int_{-1/2}^{1/2} \boldsymbol{f}_a^{\text{tr}}\boldsymbol{A}_2\boldsymbol{u}_{11}\mathrm{d}x},$$

$$-r_3 = \frac{\int_{-1/2}^{1/2} \boldsymbol{f}_a^{\text{tr}}\left[\boldsymbol{B}_{c1}^{(1)}\boldsymbol{g}_{31} + T_2^*\boldsymbol{A}_2\boldsymbol{g}_{21} - \frac{T_2^*}{T_2}\boldsymbol{B}_{c2}\boldsymbol{g}_{21}\right]\mathrm{d}x}{T_2\int_{-1/2}^{1/2} \boldsymbol{f}_2^{\text{tr}}\boldsymbol{A}_2\boldsymbol{u}_{11}\mathrm{d}x}.$$